Based on K. H. Rosen: Discrete Mathematics and its Applications.

Lecture 11: Matrix arithmetic. Section 2.6

1 Matrix arithmetic

Definition 1. A matrix is a rectangular array of numbers. A matrix with m rows and n columns is called an $m \times n$ matrix. The plural of matrix is matrices. A matrix with the same number of rows as columns is called square. Two matrices are equal if they have the same number of rows and the same number of columns and the corresponding entries in every position are equal.

Definition 2. Let m and n be positive integers and let

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

The (i, j)-th element or entry of A is the element $a_{i,j}$, that is, the number in the ith row and jth column of A. A convenient shorthand notation for expressing the matrix A is to write $A = [a_{i,j}]$, which indicates that A is the matrix with its (i, j)th element equal to $a_{i,j}$.

Definition 3. Let A be an $m \times k$-matrix and B be a $k \times n$-matrix. The product of A and B, denoted by AB, is the $m \times n$-matrix with its (i, j)th entry equal to the sum of the products of the corresponding elements from the ith row of A and the jth column of B. In other words, if $AB = [c_{i,j}]$, then

$$c_{i,j} = a_{i,1}b_{1,j} + \cdots + a_{i,k}b_{k,j}.$$

Example 4. Let $A = \begin{bmatrix} 3 & -2 & 1 \\ 8 & -7 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 6 \\ 2 & 3 \\ 1 & 1 \end{bmatrix}$. The matrix A is an 2×3-matrix and B is a 3×2-matrix, hence, we are allowed to do $A \cdot B$ and the result should be a 2×2-matrix:

$$A \cdot B = \begin{bmatrix} 3 & -2 & 1 \\ 8 & -7 & -3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 6 \\ 2 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} * & * \\ * & * \end{bmatrix}$$

Actually we did the scalar product of the vectors

$(3, -2, 1)\cdot(1, 2, 1) = 3(1)-2(2)+1(1) = 0$ $\quad$ $(3, -2, 1)\cdot(6, 3, 1) = 3(6)-2(3)+1(1) = 13$

$(8, -7, -3)\cdot(1, 2, 1) = 8(1)-7(2)-3(1) = 9$ $\quad$ $(8, -7, -3)\cdot(6, 3, 1) = 8(6)-7(3)-3(1) = 24$

And the answer is $\begin{bmatrix} 0 & 13 \\ -9 & 24 \end{bmatrix}$.

Remark 5. Matrix multiplication is not commutative. That is, if A and B are two matrices, it is not necessarily true that AB and BA are the same. In fact, it may be that only one of these two products is defined. For instance, if A is 2×3 and B is 3×4, then AB is defined and is 2×4; however, BA is not defined since it is not possible to multiply a 3×4 matrix and a 2×3 matrix. Even when both AB and BA are both defined and of the same size, it is not true that $AB = BA$

$$\begin{bmatrix} 3 & -2 \\ 6 & -7 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & -2 \\ 6 & -7 \end{bmatrix} = \begin{bmatrix} 9 & -9 \\ 9 & -9 \end{bmatrix}$$

We say that the product of matrices **is not a commutative operation**.

However, we keep some good properties:

(1) $A \cdot (B + C) = A \cdot B + A \cdot C$ and $(B + C) \cdot A = B \cdot A + C \cdot A.$

(2) $A \cdot (B \cdot C) = (A \cdot B) \cdot C.$

(3) For any square matrix $A_{n \times n}$ of order n we have $A \cdot I_n = I_n \cdot A = A$,

where I_n represents the identity matrix is the matrix of order n

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}_{n \times n} .$$

Definition 6. Let $A = [a_{i,j}]$ be an $m \times n$ matrix. The transpose of A, denoted by A^t, is the $n \times m$ matrix obtained by interchanging the rows and columns of A. In other words, if $A^t = [b_{i,j}]$, then $b_{i,j} = a_{j,i}$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$.

Example 7. For $A = \begin{bmatrix} 3 & -2 & 1 \\ 8 & -7 & -3 \end{bmatrix}$, the tranpose $A^T = \begin{bmatrix} 3 & 8 \\ -2 & -7 \\ 1 & -3 \end{bmatrix}$

The powers of square matrices can be defined. When A is an $n \times n$ matrix, we have

$$A^0 = I_n, \qquad A^r = \underbrace{A \cdots A}_{r \text{ times}}$$

1.1 Zero-one matrices

A matrix all of whose entries are either 0 or 1 is called a zero–one matrix.

Definition 8. Let $A = [a_{i,j}]$ and $B = [b_{i,j}]$ be $m \times n$ zero–one matrices. Then the **join** of A and B is the zero–one matrix with (i, j)th entry

$$a_{i,j} \vee b_{i,j} = \begin{cases} 1 & \text{if } a_{i,j} = 1 \text{ or } b_{i,j} = 1 \\ 0 & \text{otherwise} \end{cases}$$

The join of A and B is denoted by $A \vee B$. The **meet** of A and B is the zero–one matrix with (i, j)th entry

$$a_{i,j} \wedge b_{i,j} = \begin{cases} 1 & \text{if } a_{i,j} = 1 \text{ and } b_{i,j} = 1 \\ 0 & \text{otherwise} \end{cases}$$

The meet of A and B is denoted by $A \wedge B$.

Definition 9. Let A be an $m \times k$-matrix and B be a $k \times n$-matrix. The Boolean product of A and B, denoted by $A \odot B$, is the $m \times n$-matrix with its (i, j)th entry $c_{i,j}$ is equal to

$$c_{i,j} = a_{i,1} \wedge b_{1,j} \vee \cdots \vee a_{i,k} \wedge b_{k,j}.$$

Example 10. For $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ we find that

$$A^{[2]} = A \odot A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad A^{[3]} = A^{[2]} \odot A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$